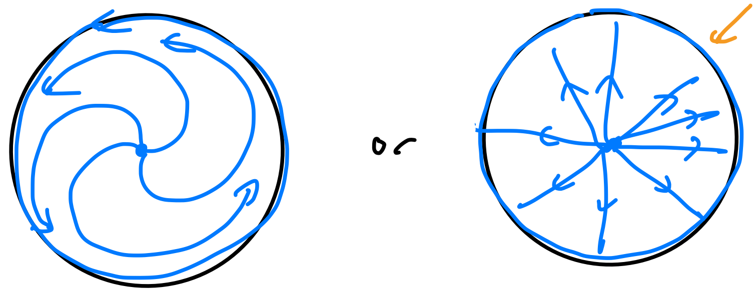
V Tight Contact Structures

a disk D in a contact 3-manifold (M,ξ) is called <u>overtwisted</u> if its characteristic foliation is



<u>exercise:</u> show (M,ξ) has one of these disks iff it has the other

We call a contact structure ξ <u>overtwisted</u> if it contains an an overtwisted disk

if ξ is not overtwisted we call it tight

let $Cont_{ot}(M) = \{ \xi \in Cont(M) : \xi \text{ overtwisted} \}$

Th$^{\underline{m}}$ IV.2 showed

$$i_*: \pi_0(Cont_{ot}(M)) \longrightarrow \pi_0(Dist(M))$$

was onto

Eliashberg showed

<u>Th$^{\underline{m}}$:</u>

$$\boxed{i_*: \pi_0(Cont_{ot}(M)) \longrightarrow \pi_0(Dist(M)) \text{ is a bijection}}$$

So two overtwisted contact structures that are homotopic are actually isotopic (and hence contactomorphic) we will not prove this theorem (complicated and uses ideas we will not develop)

Are there any tight contact structures?

Th$^{\underline{m}}$ (Bennequin 1983)

$\boxed{\text{the standard contact structure on } \mathbb{R}^3 \text{ (and } S^3\text{) is tight}}$

this theorem was the birth of contact topology and Bennequin's proof showed tight contact structures "see" subtle info about topology!

Th$^{\underline{m}}$ first proved by

Bennequin using braid theory

then by Eliashberg-Gromov using holomorphic curves

then by Kronheimer-Mrowka using Seiberg Witten theory

then by Giroux using convex surfaces

then by Ozsváth-Szabó using Heegaard Floer theory

Bennequin proved the theorem by showing for any Legendrian knot L in $(\mathbb{R}^3, \xi_{std})$ and any surface Σ with $\partial\Sigma = L$ we have

$$\boxed{tb(L) + |r(L)| \leq -\chi(\Sigma) = 2g(\Sigma) - 1}$$

note: for an overtwisted disk D we have

$$tb(\partial D) = 0$$

so

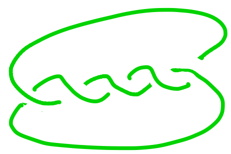
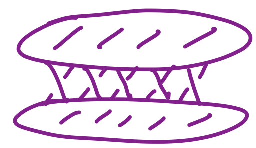
$$tb(\partial D) + |r(\partial D)| \geq 0$$

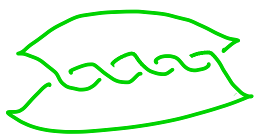
but $-\chi(D) = -1$ ⊠

so ξ_{std} is tight!

Eliashberg later showed the above inequality holds in <u>any</u> tight contact structure (we will prove this later)

<u>example:</u>

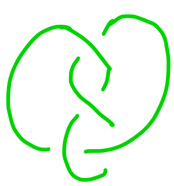
1)  bounds a genus 2 surface 

but  has $tb = 3$ $r = 0$

so $tb + |r| = 3 = 2(3) - 1$

so knot does not bound genus one surface!

2) inequality not always sharp

 figure 8 has genus 1

but can show its maximal tb is -3 with $r = 0$

so $tb + |r| = -3 < 1$

<u>lemma 1:</u>

Suppose (M, ξ) is a contact manifold and $\tilde{M}$ a covering space of M. If the pull-back $\tilde{\xi}$ of ξ to $\tilde{M}$ is tight, then so is ξ.

Proof: suppose D is an overtwisted disk in (M, ξ)

and $i: D \to M$ is the inclusion map

then the lifting criterion for covers says i lifts to

$$\begin{tikzcd} & \tilde{M} \arrow[d, "\pi"] \\ D \arrow[ur, "\tilde{i}"] \arrow[r, "i"'] & M \end{tikzcd}$$

exercise: $\tilde{i}$ an embedding and D is overtwisted disk in $(\tilde{M}, \tilde{\xi})$ $\quad \varnothing$

examples:

1) let ξ_{std} be the standard contact structure on S^3 (i.e. complex tangencies to unit $S^3 \subset \mathbb{C}^2$)

note: if $u \in S^1$ a unit complex number and $(z_1, z_2) \in S^3$ then $(u z_1, u^k z_2) \in S^3$

if u_p a primitive p^{th} root of unity in S^1

and q an integer rel. prime to p

then $S^3 \to S^3 : (z_1, z_2) \longmapsto (u_p z_1, u_p^q z_2)$

generates a free $\mathbb{Z}/p\mathbb{Z}$ action on S^3

let $L(p,q) = S^3 / \text{action}$

we call $L(p,q)$ a <u>lens space</u>

since ξ_{std} on S^3 is invariant under $\mathbb{Z}/p\mathbb{Z}$ action

we get a tight contact str. on $L(p,q)$ by lemma!

2) recall
$$\xi = \ker(\cos 2\pi z\, dx + \sin 2\pi z\, dy)$$
is equivalent to the standard contact structure on $\mathbb{R}^3$

the diffeomorphisms
$$f_1(x,y,z) = (x+1, y, z)$$
$$f_2(x,y,z) = (x, y+1, z)$$
$$f_3(x,y,z) = (x, y, z+n)$$

generate a free $\mathbb{Z}^3$ action on $\mathbb{R}^3$ with quotient T^3

ξ is invariant under this action so induce a contact structure ξ_n on T^3

from lemma 1 the ξ_n are tight!

exercise:

1) Show all the ξ_n are homotopic as plane fields

2) Show all ξ_n are distinct!

Hint: hard, we will do later

Remark: this shows there is a big difference between tight and overtwisted contact structures

overtwisted: any homotopy class of plane field has a unique overtwisted str.

tight: a homotopy class can have infinitely many tight structures

it turns out, that homotopy classes, or even 3-manifolds, might not have any tight contact structures

Th$^{\text{m}}$ (E-Honda):

$\boxed{\text{The Poincaré homology sphere with reversed orientation does not have a tight contact str!}}$

we might prove this later

here is another way to get tight contact structures

a 2-form ω on a $2n$-dimensional manifold X is called a symplectic form if

- $d\omega = 0$
- ω^n is a volume form

(X,ω) called a symplectic manifold

note X is oriented by ω^n

we say (X,ω) is a weak symplectic filling of a contact 3-manifold (M,ξ) if

- X compact
- $\partial X = M$ (as oriented manifolds, recall ξ orients M)
- $\omega|_\xi \neq 0$

Thm (Eliashberg, Gromov):

$$\boxed{\text{if } (M,\xi) \text{ is weakly symplectically fillable then } \xi \text{ is tight}}$$

exercise: Show unit $B^4 \subset \mathbb{C}^2$ with $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ is a weak symplectic filling of (S^3, ξ_{std})

(we do this later, but try using definition above)

We say (X,ω) has convex boundary if $\exists$ a vector field v defined near ∂X such that

- v points out of ∂X
- $\mathcal{L}_v \omega = \omega$ (flow of ω expands ω)

let $\alpha = \iota_v \omega$

note: α is a contact form on ∂X

indeed $\mathcal{L}_v\omega = d\iota_v\omega + \iota_v d\omega = d\iota_v\omega = d\alpha$

so $\alpha \wedge d\alpha = (\iota_v\omega)\wedge\omega = \frac{1}{2}\iota_v(\omega\wedge\omega)$

is a volume form on ∂X

$\therefore$ α contact form

we say (X,ω) is a strong symplectic filling of (M,ξ) if

∂X convex with vector field v and

(M,ξ) contactomorphic $(\partial X, \ker(\iota_v\omega))$

note: since ω is positive on $\ker \iota_v\omega$ we see

a strong filling is a weak filling

example:

on $\mathbb{C}^2$ consider $\omega = dx_1\wedge dy_1 + dx_2\wedge dy_2$

and $v = \frac{1}{2}\left(x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2}\right)$ radial field

$\mathcal{L}_v\omega = d\iota_v\omega + \iota_v d\omega = \frac{1}{2}d(x_1dy_1 - y_1dx_1 + \dots)$

$= \frac{1}{2}(2dx_1\wedge dy_1 + \dots) = \omega$

so if B^4 unit ball, ∂B^4 convex

earlier we saw $\iota_v\omega$ gives ξ_{std} on S^3

so (S^3, ξ_{std}) is strongly filled by (B^4,ω)

we discuss one more type of fillability

Suppose X is a complex manifold

as discussed earlier the complex structure induces a complex multiplication on each tangent space

so we get a bundle map
$$J : TX \to TX$$
s.t. $J^2 = -id_{TX}$

this is called an almost complex structure

given a function $\phi : X \to \mathbb{R}$ consider the 1-form
$$\lambda(v) = -d\phi(Jv)$$
we say ϕ is J-convex, also called (strictly) pluri-subharmonic, if
$$d\lambda(v, Jv) > 0 \quad \text{for all non-zero } v$$

a complex manifold (X, J) is called Stein if $\exists$ a

- J-convex function $\phi : X \to \mathbb{R}$ that is
- bounded below and
- proper (pre image of compact sets is compact)

Stein manifolds are important in complex analysis
but for us they are important since
$$\omega_\phi = -d(d\phi \circ J)$$
is a symplectic form

exercise: if (X, J) is a Stein manifold and x a regular value of ϕ
then $M = \phi^{-1}(x)$ is a smooth manifold
$-d\phi \circ J$ is a contact form on M
and $(X_x = \phi^{-1}((-\infty, x]), \omega_\phi)$ is a strong symplectic filling of $(M, \ker(-d\phi \circ J))$

we call X_κ a Stein domain

and say $(M, \ker(-d\phi \circ J))$ is Stein fillable

note: Stein fillable $\Rightarrow$ strongly fillable

$\Rightarrow$ weakly fillable

$\Rightarrow$ tight

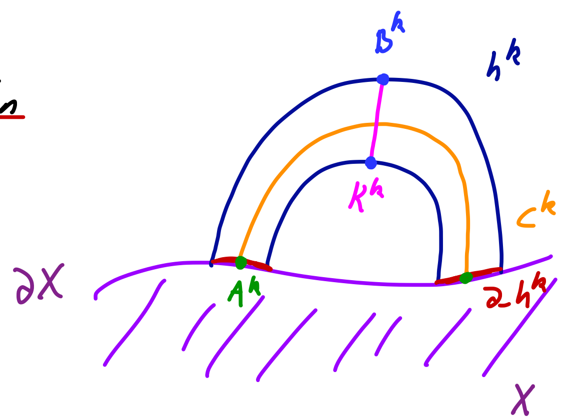
none of the reverse implications are true

Eliashberg and Gompf gave a systematic way to costruct Stein domains

to describe it, we need handlebodies

an n-dimensional k-handle is

$$h^k = D^k \times D^{n-k}$$



Set $\partial_- h^k = (\partial D^k) \times D^{n-k}$ attaching region

$\partial_+ h^k = D^k \times (\partial D^{n-k})$

$A^k = (\partial D^k) \times \{0\}$ attaching sphere

$C^k = D^k \times \{0\}$ core

$K^k = \{0\} \times D^{n-k}$ co-core

$B^k = \{0\} \times (\partial D^{n-k})$ belt sphere

given an n-manifold X and an embedding

$$\phi: \partial_- h^k \to \partial X$$

we attach h^k to X by forming the identification space

$$X \cup_\phi h^k = X \amalg h^k \big/ (x \in \partial_- h^k) \sim (\phi(x) \in \partial M)$$

exercise: if $\phi_0, \phi_1 : \partial_- h^k \to \partial X$ are isotopic then

$X \cup_{\phi_0} h^k$ diffeomorphic to $X \cup_{\phi_1} h^k$

note: $\phi(A)$ is a $(k-1)$-knot in ∂X

$\phi(\partial_- h^k)$ gives a framing to $\phi(A)$

exercise: isotopy type of ϕ is determined by

1) isotopy type of $\phi(A)$ and
2) framing on $\phi(A)$

also sho framings on $\phi(A)$ are in one-to-one correspondence with

$$\pi_{k-1}(O(n-k))$$

or $\pi_{k-1}(SO(n-k))$ if $k>1$

the exercise says one may attach a k-handle to X by specifying a knotted $(k-1)$-sphere in ∂X and a framing on sphere

example:

1) $\partial_- h^0 = \emptyset$ so attaching 0-handle to X is just disjoint union

$$X \amalg D^n$$

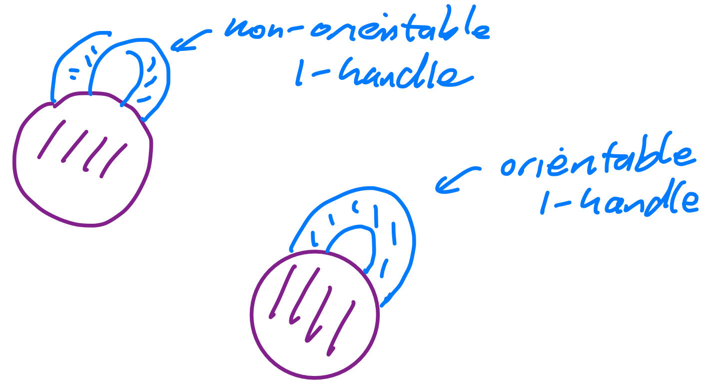
2) attaching sphere for 1-handle is S^0

framing in $\pi_0(O(n)) = \mathbb{Z}/2\mathbb{Z}$

on framing gives non-orientable manifold

so if we want an oriented manifold then one choice for framing

eg. 2-dimensions



A <u>handle decomposition</u> of an n-manifold M is a sequence of manifolds $M_0, M_1, \dots M_\ell$ such that

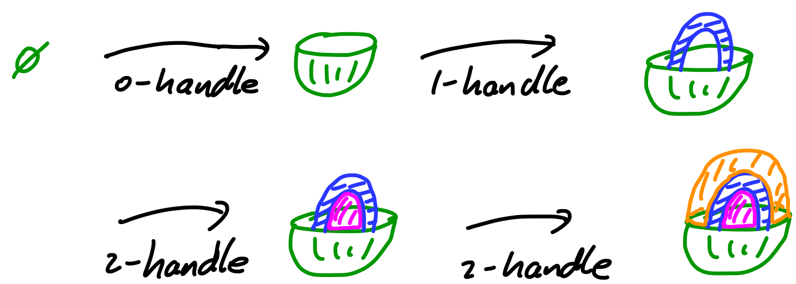
1) $M_0 = \emptyset$ and $M_\ell \cong M$
2) M_{i+1} is obtained from M_i by a k-handle attachment for some k

<u>example</u>:

handle decompositions of S^2

1)

2)



<u>Fact</u>: every smooth manifold has a handle decomposition

<u>exercise</u>: If X' is obtained from X^n by attaching a k-handle, then

$$\partial X' = (\partial X - (\text{nbhd } S^{k-1})) \cup (D^k \times S^{n-k-1})$$

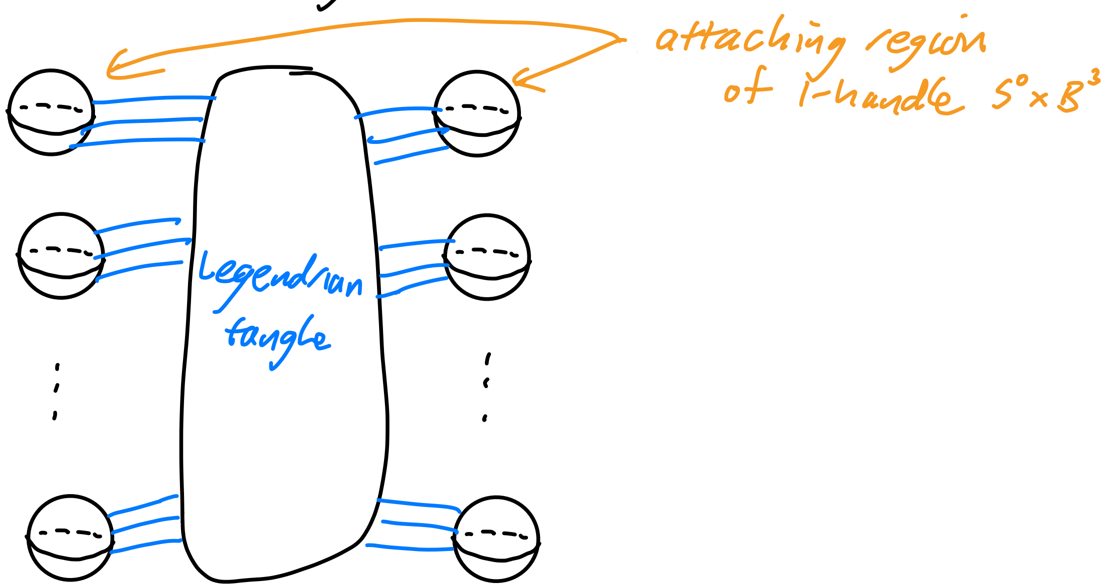
in particular attaching a 1-handle changes the boundary by # with $S^1 \times S^{n-2}$

in dimension 4, attaching a 2-handle changes the boundary by removing nbhd of a knot and gluing in a solid torus exercise: If $K \subset \partial X^4$ is null-homologous and we attach a 2-handle with framing n then boundary changes by n-Dehn surgery on K

Eliashberg showed any Stein domain can be built by

- attaching some number of 1-handles to B^4 (note $\partial = \#_k S^1 \times S^2$ and has a tight contact structure on it)
- attaching 2-handles to Legendrian knots with framing one less than the contact framing

Gompf showed you can arrange the above to look like



and formulas for Thurston-Bennequin invariant and rotation number for Legendrians in $(\mathbb{R}^3, \xi_{std})$ work here too

Th^{m} (Eliashberg 1990, Gompf 1998):

a 4-manifold X has the structure of a Stein domain

$\Leftrightarrow$

X has a handle decomposition with one 0-handle some number of 1-handles and 2-handles attached to Legendrian knots with framing $(tb-1)$.

Moreover, the 1st Chern class of X evaluated on the 2-handle h attached to L is

$$\langle c_1(X), h_i \rangle = r(L_i)$$

Remarks:

1) complex manifolds (X^{2n}, J) have Chern classes

$$c_k(X) \in H^{2k}(X) \qquad k = 1, \dots, n$$

we do not need to know exactly what they are just that they are invariants of complex manifolds

2) note attaching a k-handle $h^k = D^k \times D^{n-k}$ is homotopy equivalent to attaching a k-cell (since $D^{n-k} \simeq pt$)

so the cellular homology chain groups are

$$C_k^{cw}(X) = \oplus_{n_k} \mathbb{Z}$$

where $n_k = \#$ of k-cells

3) for a Stein domain X with ξ the contact structure induced on ∂X we have

$$TX|_{\partial X} = \xi \oplus \mathbb{C}$$

so $c_1(X) = c_1(\xi) \oplus c_1(\mathbb{C}) = c_1(\xi)$

$c_1(\xi)$ is called the Euler class of ξ

and is an invariant of ξ denoted $e(\xi)$

from the formula for $c_1(X)$ we see if (M,ξ) is

the boundary of (X,J) constructed by attaching

2-handles to $L_1, \dots, L_k$ then

Poincaré Dual

meridian to L_i

$$e(\xi) = \sum r(L_i)\, P.D.(\mu_i)$$

4) when attaching a Stein 2-handle to a

Legendrian knot L, on the boundary

we do $tb(L)-1$ Dehn surgery and get

a contact structure on the resulting

manifold

we call the chang on the boundary

the result of <u>Legendrian surgery on L</u>

that is, if L a Legendrian knot in (M,ξ)

then Legendrian surgery on L is the result of

removing a nbhd N of L from M and

gluing in a solid torus with meridian

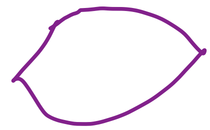
mapping to the contact framing -1

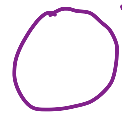
($tb-1$ if L null-homologous)

and extending $\xi|_{M-N}$ to a tight contact

structure on the surgery torus

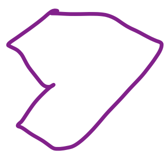
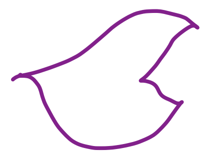
example:

1) Legendrian surgery on 

is the manifold  -2 $= L(2,1)$

so $L(2,1)$ has a tight contact structure

this is the one constructed earlier

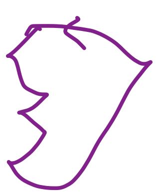
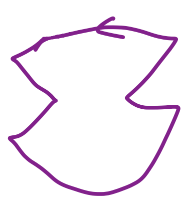
2)  and 

give $L(3,1)$ with Euler class

$$e = \pm P.D.[\mu]$$

in $H^2(L(3,1)) = \mathbb{Z}/3\mathbb{Z} = 0, \underline{1}, \underline{2}$

so distinct contact structures!

3)   

ξ_1 ξ_2 ξ_3

are contact structures on $L(4,1)$ with

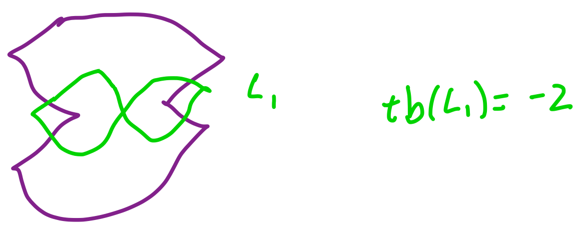
$$e(\xi_i) = \begin{cases} 2 & i=1 \\ 0 & 2 \\ -2 & 3 \end{cases}$$

ξ_2 different from ξ_1, ξ_3

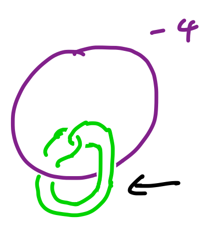
but ξ_1, ξ_3 have same Euler class in $H^2(L(4,1)) = \mathbb{Z}/4\mathbb{Z}$
(we will see below ξ_1, ξ_3 are different,
but $\xi_1 = \xi_3$ with different orientations)

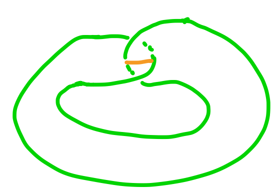
ξ_1, ξ_3 but pull-back to ξ_{std} on S^3 (i.e. we constructed above)

ξ_2 pulled back to S^3 is overtwisted
to see this consider L' in $(L(4,1), \xi_2)$



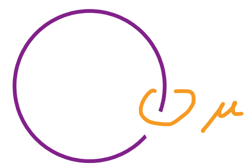
smoothly we see





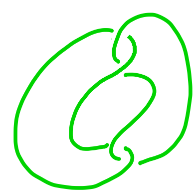
exercise: framing induce by singular disk is -2 (so contact and singular disk framing same)

note: $\pi_1(L(4,1)) = \mathbb{Z}/4\mathbb{Z}$ is generated by



so μ unwraps in any cover

in 2 fold cover singular disk lifts to



two embedded disk

and contact framing = disk framing

so $tb(\text{this unknot}) = 0$

$\cancel{\exists}$ Bennequin $\leq$ so cover is overtwisted

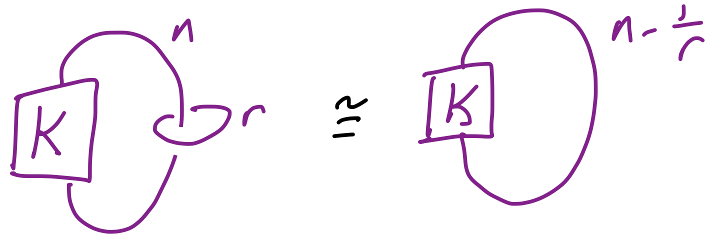
note: this means this tight contact structure on $L(4,1)$ could not have come from (S^3, ξ_{std}) like the ones above did!

a contact structure ξ on M is called

universally tight if the pull-back to the universal cover of M is tight

virtually overtwisted if the pull-back to a finite cover is overtwisted exercise:

1) show



called a slam dunk

where $n \in \mathbb{Z}$, $r \in \mathbb{Q}$

2) any rational number can be written

$$r = a_0 - \cfrac{1}{a_1 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}$$

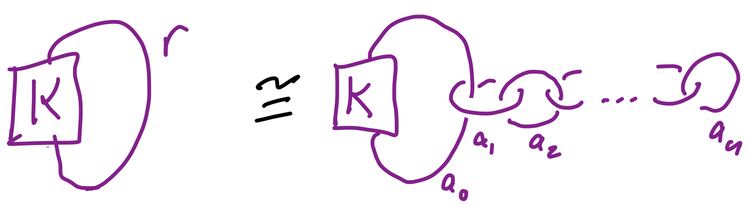
called the contenued fraction of r

with $a_i \in \mathbb{Z}$ and $a_i \leq -2$ for $i > 0$

we write this $[a_0; a_1, \dots a_n]$

eg $-\frac{40}{11} = -4 - \cfrac{1}{-3 - \cfrac{1}{-4}} = -4 - \cfrac{1}{-\frac{11}{4}} = -4 + \frac{4}{11}$

3) so if $r = [a_0; a_1 \dots a_n]$ then

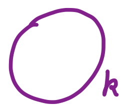


lemma 2:

> If $-p/q = [a_0, \dots, a_n]$ with all $a_i \leq -2$, then
>
> $L(p,q)$ has at least $|(a_0+1) \cdots (a_n+1)|$
>
> tight contact structures

Proof: $L(p,q) =$ 

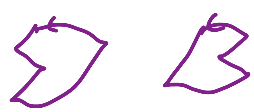
$\cong$ 

note: given 

$k \leq -2$

if we want to attach a Stein 2-handle to affect this surgery we need a Legendrian representative with $tb = k+1$


$k=-2$


$k=-3$

in general there are $|k+1|$ Legendrian knots with $tb = k+1$ and all have distinct rotation numbers

so there are $|(a_0+1)\cdots(a_n+1)|$ ways to realize

as a Legendrian link so that attaching Stein 2-handles will give $L(p,q)$

exercise: when p odd show these structures are all distinct because they have different Euler classes but this is not true for (most) p even.

we use a theorem of Lisca-Matić (that uses Seiberg-Witten theory)

Thm:

> Suppose X has two Stein structures built with one 0-handle and some 2-handles
> Denote their almost complex structures J_1, J_2.
> If $c_1(J_1) \neq c_1(J_2)$ then the contact structures induced on ∂X are different

this completes the lemma $\blacksquare$

Remark: later we will see that the contact strs constructed above on $L(p,q)$ are the only tight contact structures on $L(p,q)$

Fact: $L(p,q)$ and $L(p,q')$ are homotopy equivalent

$$\Leftrightarrow$$

$$qq' \equiv \pm n^2 \bmod p$$

note: $L(7,2)$ and $L(7,1)$ are homotopy equivalent since $2 \equiv 4^2 \bmod 7$

but $L(7,1)$ has $|-7+1| = 6$ tight contact structures

$-\frac{7}{2} = -4 - \frac{1}{-2}$ $\quad$ $L(7,2)$ has $|-4+1|\,|-2+1| = 3$ " "

so $L(7,1)$, $L(7,2)$ are <u>not</u> diffeomorphic

and tight contact structures are sensitive enough to detect this!